Modular Termination of Graph Transformation[1]

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[1]In *Graph Transformation, Specifications, and Nets: In Memory of Hartmut Ehrig*. LNCS 10800, Springer, 2018

Termination

- Impossibility of any infinite sequence

$$G_0 \Rightarrow_{\mathcal{R}} G_1 \Rightarrow_{\mathcal{R}} G_2 \Rightarrow_{\mathcal{R}} \dots$$

  given a set $\mathcal{R}$ of DPO graph transformation rules

- Guarantees that the non-deterministic strategy

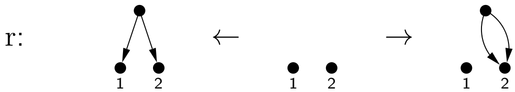
  apply rules as long as possible

  returns a result on all graphs

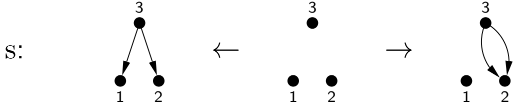
- Corresponds to program termination in conventional programming languages: program halts on all inputs

- Undecidable in general

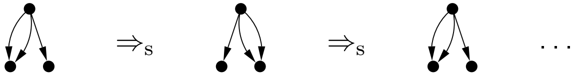
One-rule examples (assuming injective matching)



Terminating: Every step $G \Rightarrow_{\mathrm{r}} H$ reduces the number of nodes whose out-edges have different targets.



Looping:



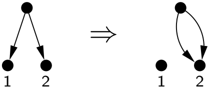
Modularity of termination

Observation

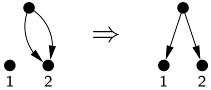
The union of terminating rule sets need not be terminating.

Example

Both

r: 

and

r^{-1}: 

are terminating but $\{\mathrm{r}, \mathrm{r}^{-1}\}$ is looping

A machine-checkable condition on rule sets such that termination of $\mathcal{R}$ and $\mathcal{S}$ implies termination of $\mathcal{R} \cup \mathcal{S}$.

Hypergraph transformation

- Directed hypergraphs with node and edge labels.
- Rules $r\colon \langle L \leftarrow K \rightarrow R \rangle$ consist of two hypergraph morphisms, where $L \leftarrow K$ is an inclusion.

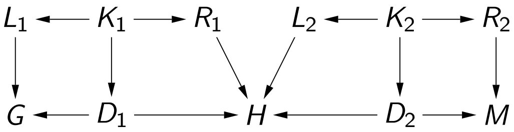
  Special case: *injective* rules where $K \rightarrow R$ is injective.
- Direct derivations $G \Rightarrow_{r,g} H$ are double-pushouts with injective match $g\colon L \rightarrow G$:

$$\begin{array}{ccccc} L & \longleftarrow & K & \longrightarrow & R \\ {\scriptstyle g}\downarrow & \text{PO} & \downarrow & \text{PO} & \downarrow \\ G & \longleftarrow & D & \longrightarrow & H \end{array}$$

- Hypergraph transformation systems $\langle \Sigma, \mathcal{R} \rangle$ consist of a signature Σ and a finite set $\mathcal{R}$ of rules over Σ.

Sequential independence

Two direct derivations



are *sequentially independent* if there are $R_1 \to D_2$, $L_2 \to D_1$ s.t.

1. $R_1 \to H = R_1 \to D_2 \to H$ and $L_2 \to H = L_2 \to D_1 \to H$
2. $R_1 \to D_2 \to M$ is injective

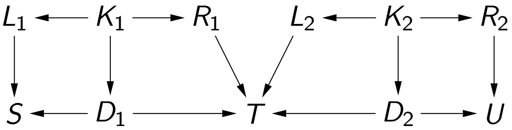
Note: 2nd condition is satisfied if $\langle L_2 \leftarrow K_2 \to R_2 \rangle$ is injective.

Theorem (Habel-Müller-P 98, Ehrig-Kreowski 76)

If $G \Rightarrow_{r_1} H \Rightarrow_{r_2} M$ are sequentially independent then there exists a graph H' such that $G \Rightarrow_{r_2} H' \Rightarrow_{r_1} M$.

Sequential critical pairs

A *sequential critical pair* consists of direct derivations

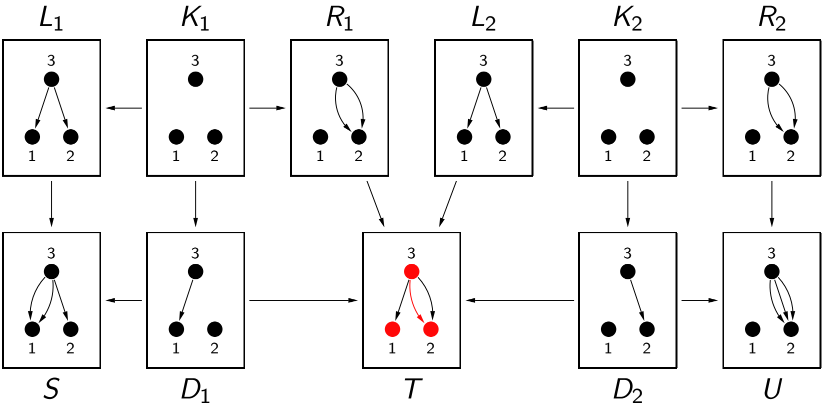


such that the following holds.

1. *Conflict:* The steps are not sequentially independent.
2. *Minimality:* $R_1 \to T \leftarrow L_2$ are jointly surjective.

Note: Finite rule sets possess, up to isomorphism, only finitely many critical pairs.

Example: sequential critical pair



- $\nexists\ (R_1 \to D_2,\ L_2 \to D_1)$ such that $R_1 \to T = R_1 \to D_2 \to T$ and $L_2 \to T = L_2 \to D_1 \to T$
- Equivalently, $h(R_1) \cap g(L_2) \neq h(K_1) \cap g(K_2)$

Main result

Theorem (Modularity of termination)

Let $\langle \Sigma, \mathcal{R} \rangle$ and $\langle \Sigma, \mathcal{S} \rangle$ be terminating systems. If there are no critical pairs of form $S \Rightarrow_{\mathcal{R}} T \Rightarrow_{\mathcal{S}} U$, then the combined system $\langle \Sigma, \mathcal{R} \cup \mathcal{S} \rangle$ is terminating.

Remark

Notice the symmetry in the statement: $\mathcal{R} \cup \mathcal{S}$ can have critical pairs of form either $\Rightarrow_{\mathcal{R}}\Rightarrow_{\mathcal{S}}$ or $\Rightarrow_{\mathcal{S}}\Rightarrow_{\mathcal{R}}$, but not of both forms.

Proof of main result

Let $\langle \Sigma, \mathcal{R} \rangle$ and $\langle \Sigma, \mathcal{S} \rangle$ be terminating systems and assume that there are no critical pairs of form $S \Rightarrow_{\mathcal{R}} T \Rightarrow_{\mathcal{S}} U$. Suppose there is an infinite derivation

$$G_1 \underset{\mathcal{R} \cup \mathcal{S}}{\Rightarrow} G_2 \underset{\mathcal{R} \cup \mathcal{S}}{\Rightarrow} G_3 \underset{\mathcal{R} \cup \mathcal{S}}{\Rightarrow} \dots$$

Because $\mathcal{R}$ and $\mathcal{S}$ are terminating, the derivation must contain infinitely many $\Rightarrow_{\mathcal{R}}$-steps and infinitely many $\Rightarrow_{\mathcal{S}}$-steps. Any two steps $G_k \Rightarrow_{\mathcal{R}} G_{k+1} \Rightarrow_{\mathcal{S}} G_{k+2}$ in the sequence must be sequentially independent: otherwise they could be restricted to a critical pair of form $S \Rightarrow_{\mathcal{R}} T \Rightarrow_{\mathcal{S}} U$. By sequential independence, the steps can be swapped such that $G_k \Rightarrow_{\mathcal{S}} G'_{k+1} \Rightarrow_{\mathcal{R}} G_{k+2}$. Thus all $\Rightarrow_{\mathcal{S}}$-steps can be pushed to the beginning of the derivation, resulting in an infinite sequence of $\Rightarrow_{\mathcal{S}}$-steps (illustration follows). This contradicts the fact that $\langle \Sigma, \mathcal{S} \rangle$ is terminating. $\square$

Proof illustration: sorting an infinite derivation

$$G_0 \Rightarrow_{\mathcal{R}} G_1 \Rightarrow_{\mathcal{R}} G_2 \Rightarrow_{\mathcal{S}} G_3 \Rightarrow_{\mathcal{R}} G_4 \Rightarrow_{\mathcal{S}} G_5 \Rightarrow \dots$$

$$\downarrow$$

$$G_0 \Rightarrow_{\mathcal{R}} G_1 \Rightarrow_{\mathcal{S}} G_2' \Rightarrow_{\mathcal{R}} G_3 \Rightarrow_{\mathcal{R}} G_4 \Rightarrow_{\mathcal{S}} G_5 \Rightarrow \dots$$

$$\downarrow$$

$$G_0 \Rightarrow_{\mathcal{S}} G_1' \Rightarrow_{\mathcal{R}} G_2' \Rightarrow_{\mathcal{R}} G_3 \Rightarrow_{\mathcal{R}} G_4 \Rightarrow_{\mathcal{S}} G_5 \Rightarrow \dots$$

$$\downarrow$$

$$G_0 \Rightarrow_{\mathcal{S}} G_1' \Rightarrow_{\mathcal{R}} G_2' \Rightarrow_{\mathcal{R}} G_3 \Rightarrow_{\mathcal{S}} G_4' \Rightarrow_{\mathcal{R}} G_5 \Rightarrow \dots$$

$$\downarrow$$

$$G_0 \Rightarrow_{\mathcal{S}} G_1' \Rightarrow_{\mathcal{R}} G_2' \Rightarrow_{\mathcal{S}} G_3' \Rightarrow_{\mathcal{R}} G_4' \Rightarrow_{\mathcal{R}} G_5 \Rightarrow \dots$$

$$\downarrow$$

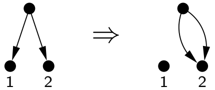
$$G_0 \Rightarrow_{\mathcal{S}} G_1' \Rightarrow_{\mathcal{S}} G_2'' \Rightarrow_{\mathcal{R}} G_3' \Rightarrow_{\mathcal{R}} G_4' \Rightarrow_{\mathcal{R}} G_5 \Rightarrow \dots$$

$$\downarrow$$

$$\vdots$$

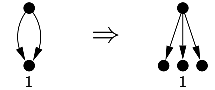
Example 1

r:



Reduces the number of nodes whose out-edges have different targets.

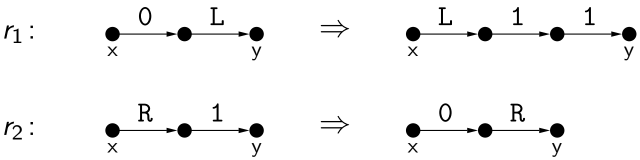
s:



Reduces the number of nodes whose out-edges have a shared target.

There is no critical pair $S \underset{\mathrm{s}}{\Rightarrow} T \underset{\mathrm{r}}{\Rightarrow} U$, hence $\{\mathrm{r}, \mathrm{s}\}$ is terminating.

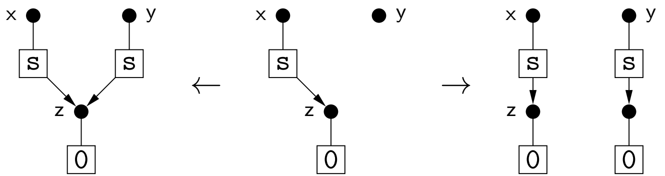
Example 2



- Shown to be terminating in [Bruggink-König-Zantema 14] by constructing a weighted type graph over the tropical semiring.
- Simple termination proof by modularity: r_1 reduces the number of 0's and r_2 reduces the number of 1's, hence both rules are terminating. There are no critical pairs of form $S \Rightarrow_{r_1} T \Rightarrow_{r_2} U$, thus $\{r_1, r_2\}$ is terminating.

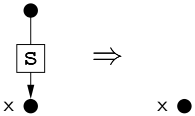
Example 3 (jungles)

c :

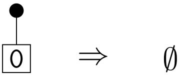


(copy rule for 0)

g_1 :



g_2 :



(garbage collection)

- Rule c reduces the value $\sum_{v \in V_G} \mathrm{indegree}(v)^2$
- Rules g_1 and g_2 are size-reducing
- There are no critical pairs of form $S \Rightarrow_{g_{1/2}} T \Rightarrow_c U$, thus $\{c, g_1, g_2\}$ is terminating

Conclusion

- *Black box-combination* of termination proofs: the proofs of the component systems need not be inspected and can be constructed using arbitrary techniques
- Condition can be mechanically checked by generating sequential critical pairs between component systems
- Applicable to arbitrary (hyper-)graph transformation systems with injective and non-injective rules

Related work

Theorem (Dershowitz, ICALP 1981)

Let $\mathcal{R}$ and $\mathcal{S}$ be terminating term-rewriting systems over some set of terms T. If $\mathcal{R}$ is left-linear, $\mathcal{S}$ is right-linear, and there is no overlap between the left-hand sides of $\mathcal{R}$ and right-hand sides of $\mathcal{S}$, then the combined system $\mathcal{R} + \mathcal{S}$ also terminates.

Future work

Theorem (Generalised result)

Let $\langle \Sigma, \mathcal{R} \rangle$ and $\langle \Sigma, \mathcal{S} \rangle$ be terminating systems. The combined system $\langle \Sigma, \mathcal{R} \cup \mathcal{S} \rangle$ is terminating if the following holds: For each critical pair of form $S \Rightarrow_{\mathcal{R}} T \Rightarrow_{\mathcal{S}} U$ there exists a derivation

$$S \overset{+}{\underset{\mathcal{S}}{\Rightarrow}} T' \overset{*}{\underset{\mathcal{R}}{\Rightarrow}} U$$

such that $\mathrm{track}_{S \Rightarrow^+_{\mathcal{S}} T' \Rightarrow^*_{\mathcal{R}} U}$ is defined for all nodes in S.

Note: The condition is mechanically checkable.

Extensions

- Rules with application conditions (e.g. NACs)
- Attributed graph transformation
- Graph programs