ICGT 2018: CoReS: A Tool for Computing Core Graphs via SAT/SMT Solvers

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Analyse the behaviour and verify the correctness of dynamically evolving systems.

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- Variable topologies
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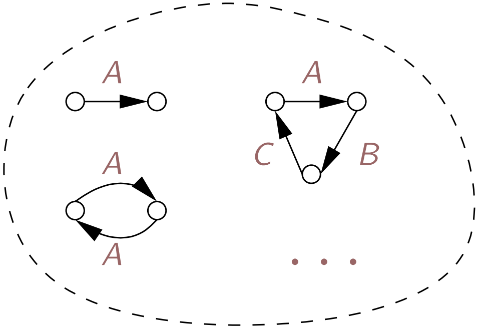
Trade-off: More complex modeling language $\rightsquigarrow$ harder analysis.

Overview

In this Talk

Specify (possibly infinite) sets of graphs by finite graphs and compute their corresponding minimal representation.



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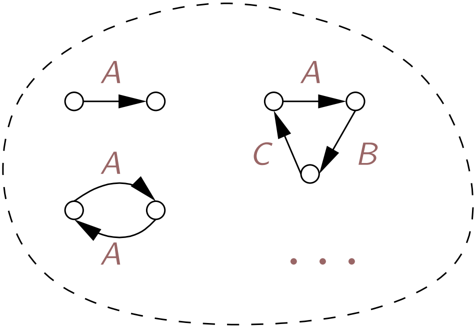

Solving a *subtask* from our predecessor paper (ICGT 2017)

Contents

Part I

Background and Preliminaries

The Basic Framework of Type Graphs

We started by studying type graphs as a specification language.

Type Graph Language

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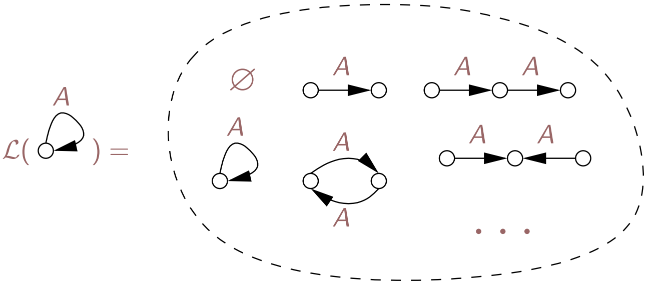

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- Other formalisms are based on type graphs (e.g., abstract graphs that use type graphs with additional annotations)
- Refine/Extend this basic formalism and analyse the properties.

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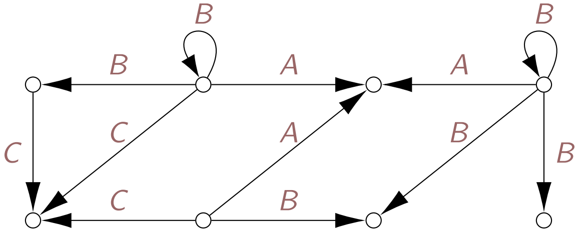
Today's aim:

Efficiently minimize the type graph without changing its language.

Minimization



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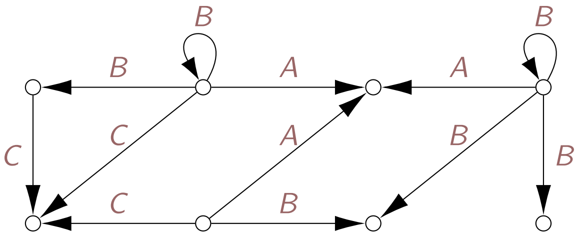

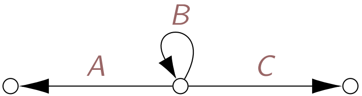

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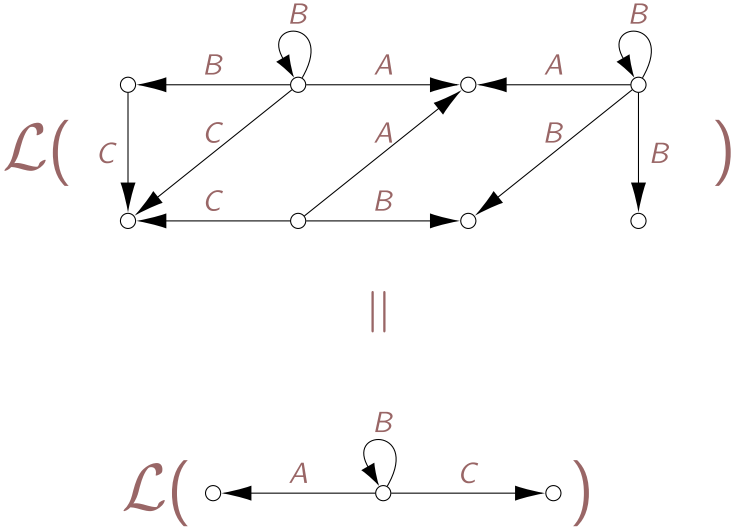

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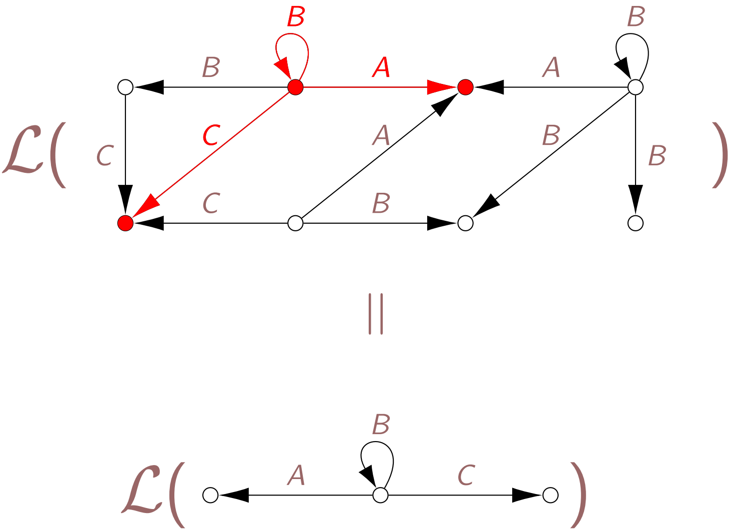

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Retracts and Core Graphs

A subgraph T' of a graph T for which there exists a morphism $\varphi\colon T \to T'$ is called a retract of T.

If a graph has no proper retracts itself, it is called core graph. (Nešetřil, Tardif).

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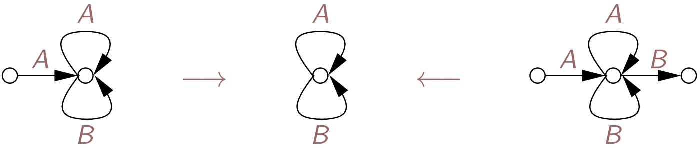


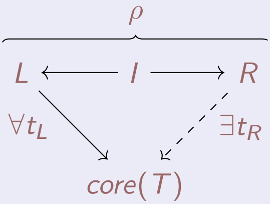
Core

Invariant Checking

Let T be a graph and $core(T)$ be its core.

Closure under rewriting

$\mathcal{L}(T)$ is closed under application of $\rho \iff$



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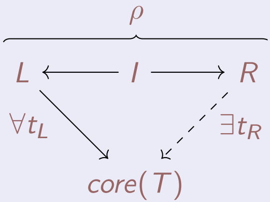

Question: How can we efficiently compute the core graph?

Part II

Core Computation via SAT/SMT Encodings

The Problem

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Retract Morphism Problem

Given a graph G. Does there exist a non-surjective endomorphism $\varphi' \colon G \to G$ with $\varphi'|_H = id_H$ where $H = img(\varphi')$?

SMT Solver

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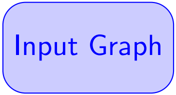
Example

```
(declare-const x Int)                      | x, y ∈ Int
(declare-const y Int)
(assert (= (- x y) (+ x (- y) 1)))         | x − y = x − y + 1
(check-sat)
```

Core Computation in a Nutshell



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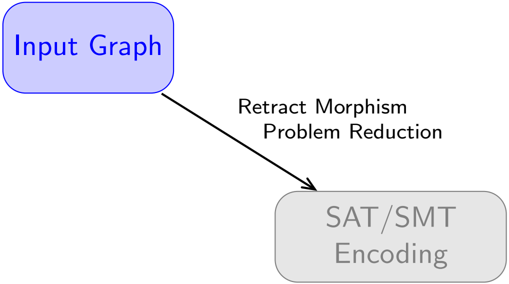

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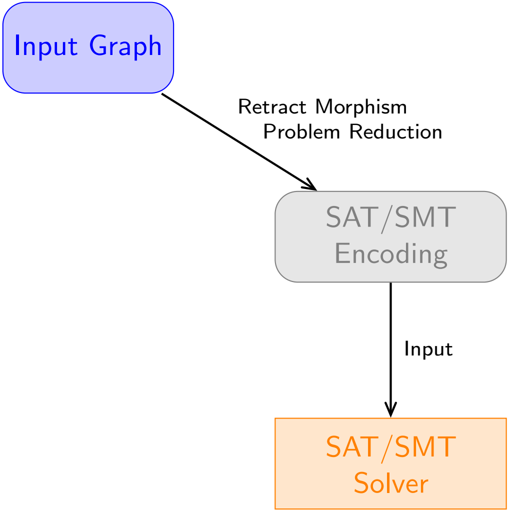

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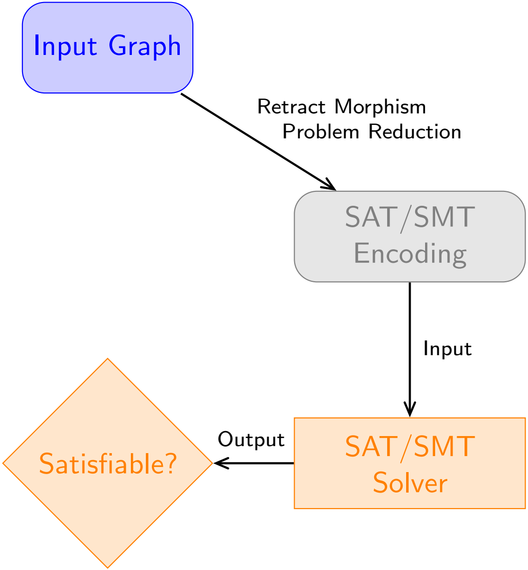

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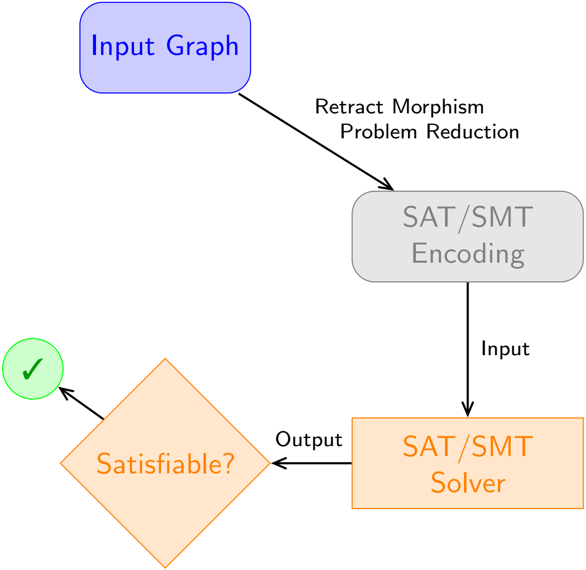

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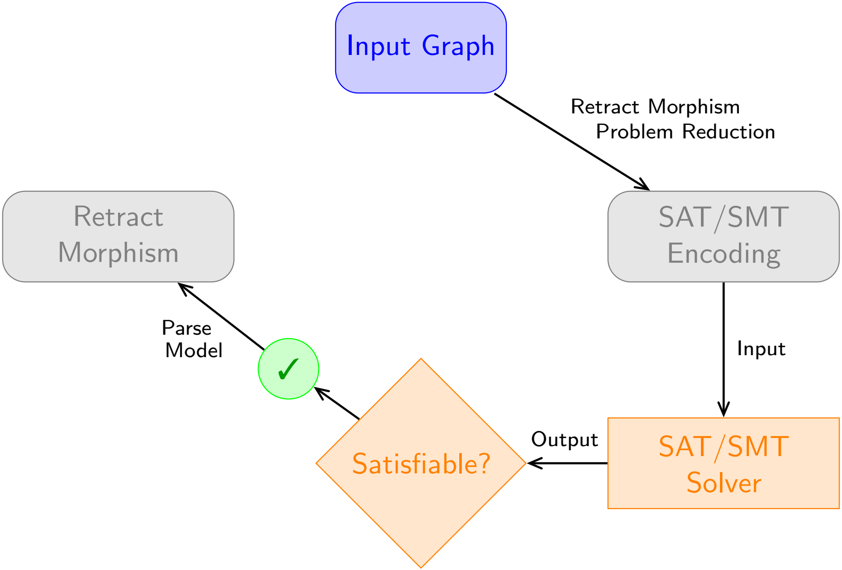

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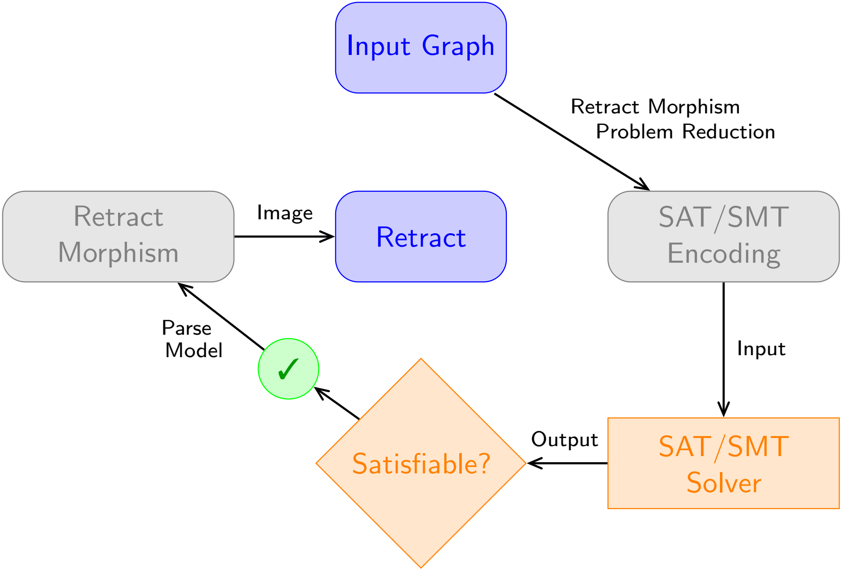

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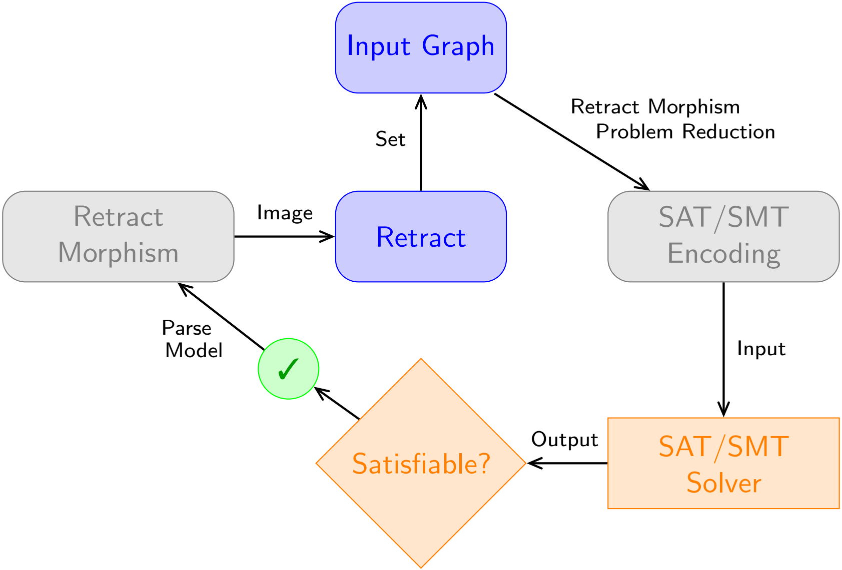

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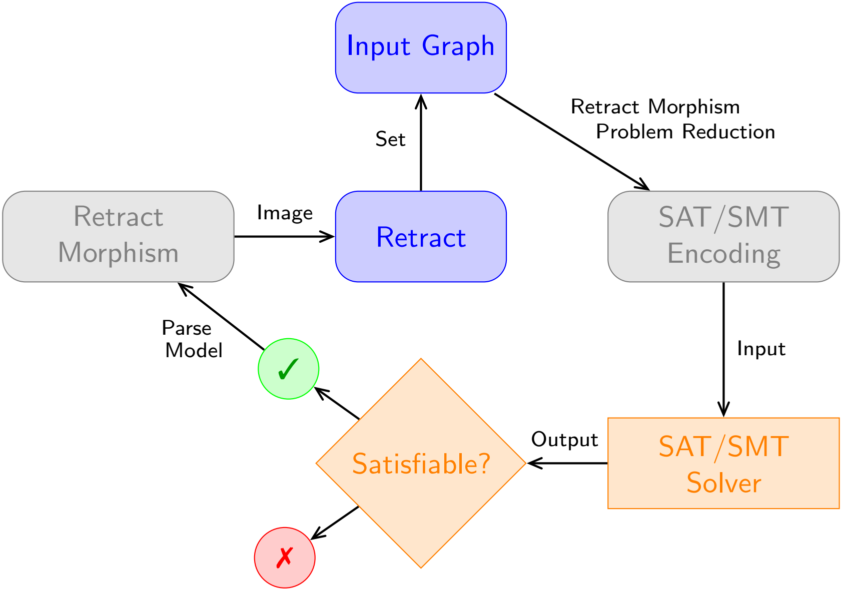

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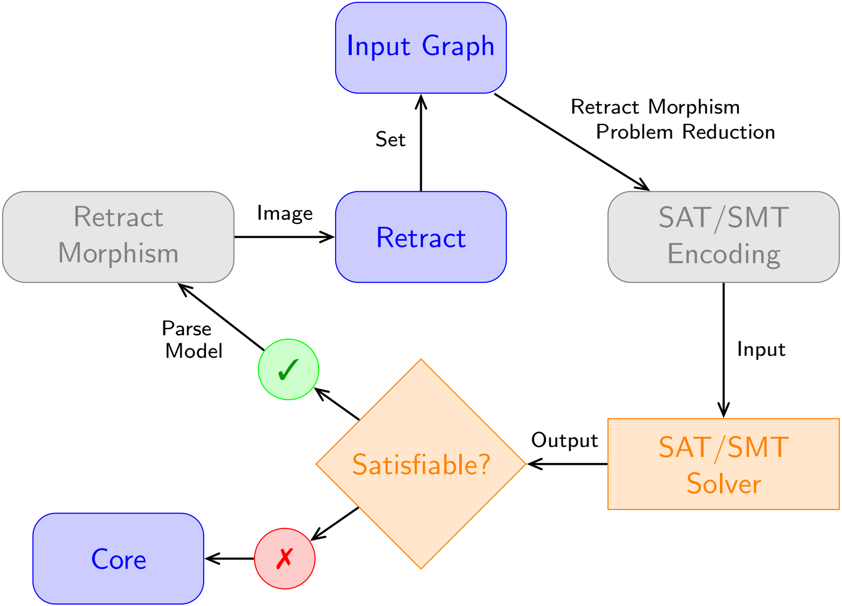

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The morphism φ needs to be structure preserving, i.e.

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3) Retract property:
The morphism φ restricted on its image is an identity morphism, i.e.

$$\varphi|_{img(\varphi)} = id_{img(\varphi)}$$

SMT-LIB2 Encoding of Retract Morphism Properties

Initialize the components of the input $G = (V, E, src, tgt, lab)$:

(declare-datatypes () ((V v1 ... vN)))	$\mid (V = \{v_1, \ldots, v_n\})$
(declare-datatypes () ((E e1 ... eM)))	$\mid (E = \{e_1, \ldots, e_m\})$
(declare-datatypes () ((L A ...)))	$\mid (\Lambda = \{A, \ldots\})$
(declare-fun src (E) V)	$\mid src\colon E \to V$
(declare-fun tgt (E) V)	$\mid tgt\colon E \to V$
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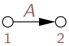
For instance the graph 

can be encoded in the following way:

(assert (= (src e1) v1))	$\mid src(e_1) = v_1$
(assert (= (tgt e1) v2))	$\mid tgt(e_1) = v_2$
(assert (= (lab e1) A))	$\mid lab(e_1) = A$

SMT-LIB2 Encoding of Retract Morphism Properties

Next, we specify the constraints for the morphism $\varphi\colon G \to G$:

1) Graph morphism property

(declare-fun vphi (V) V)	$\mid \varphi_V\colon V \to V$
(declare-fun ephi (E) E)	$\mid \varphi_E\colon E \to E$
(assert (forall ((e E)) (= (src (ephi e)) (vphi (src e)))))	$\mid src(\varphi_E(e)) = \varphi_V(src(e))$
(assert (forall ((e E)) (= (tgt (ephi e)) (vphi (tgt e)))))	$\mid tgt(\varphi_E(e)) = \varphi_V(tgt(e))$
(assert (forall ((e E)) (= (lab (ephi e)) (lab e))))	$\mid lab(\varphi_E(e)) = lab(e)$

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2) Subgraph property

(assert (exists ((v1 V)) not(exists ((v2 V)) (= v1 (vphi v2)))))	$\mid \exists v_1 \in V \neg \exists v_2 \in V : v_1 = \varphi_V(v_2)$

SMT-LIB2 Encoding of Retract Morphism Properties

We need to specify that the retract property $\varphi|_{img(\varphi)} = id_{img(\varphi)}$ holds. We rephrase this requirement in the following way:

$$\forall x \in G\Big((\exists y \in G \ (\varphi(y) = x)) \implies \varphi(x) = x \Big)$$

Every element in the image of φ is part of the retract and therefore always has to be mapped to itself.

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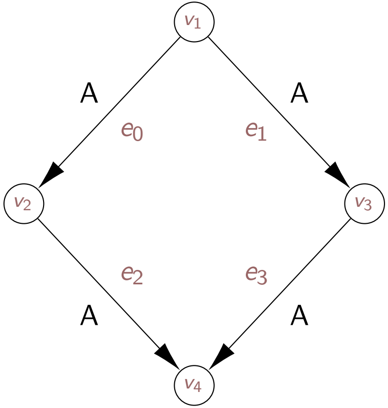
3) Retract property

```
(assert (forall ((v1 V)) (=> (exists ((v2 V)) (= v1 (vphi v2))) (= v1 (vphi v1)))))
(assert (forall ((e1 E)) (=> (exists ((e2 E)) (= e1 (ephi e2))) (= e1 (ephi e1)))))
```

Example Graph



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Our set of atomic propositions $\mathcal{A}$ has size $|\mathcal{A}| = |V \times V|$.

For a pair of nodes $(x, y) \in V \times V$ we use `Ax-y` with

$$\mathcal{A} \ni \texttt{A}x\texttt{-}y \equiv \texttt{true} \text{ iff } \varphi_V(x) = y \text{ holds.}$$

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Additional requirement

$$\bigwedge_{x \in V} \bigvee_{y \in V} \Big(\texttt{A}x\texttt{-}y \wedge \big(\bigwedge_{z \in V \setminus \{y\}} \neg \texttt{A}x\texttt{-}z \big) \Big) \quad | \; \forall x \exists! y \; \varphi_V(x) = y$$

SAT Encoding of Retract Morphism Properties

1) Graph morphism property

$$\bigwedge_{e \in E} \bigvee_{e' \in E_{lab(e)}} \Big((\mathtt{A}src(e)\text{-}src(e')) \wedge (\mathtt{A}tgt(e)\text{-}tgt(e')) \Big)$$

2) Subgraph property

$$\bigvee_{x \in V} \Big(\bigwedge_{y \in V} \neg \mathtt{A}y\text{-}x \Big) \qquad | \exists x \forall y \ \varphi(y) \neq x$$

3) Retract property

$$\bigwedge_{x \in V} \Big(\big(\bigvee_{y \in V} \mathtt{A}y\text{-}x \big) \Rightarrow \mathtt{A}x\text{-}x \Big) \qquad | \ \varphi_{|H} = id_H$$

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The derivation of the formulas above is given in our paper.

Part III

CoReS

(Computation of Retracts encoded SAT/SMT)

Experiments

The encodings were tested on 125 random graphs consisting of

- a fixed number of nodes $|V|$.
- a fixed number of available edge labels $|\Lambda|$.
- a fixed probability ρ for an edge to exist.

SAT (Limboole) vs SMT (Z3)

		$\rho \cdot \lvert V\rvert \cdot \lvert\Lambda\rvert$									
		0.5		0.8		1.0		1.2		1.5	
$\lvert V\rvert$	$\lvert\Lambda\rvert$	SAT	SMT	SAT	SMT	SAT	SMT	SAT	SMT	SAT	SMT
16	1	.075	.116	.078	.344	.078	.733	.071	1.17	.070	3.01
	2	.067	.155	.096	.463	.080	1.12	.079	2.11	.078	4.21
	3	.063	.172	.100	.548	.074	1.14	.071	2.02	.073	4.09
32	1	.301	.620	.306	4.58	.396	12.4	.424	27.4	.500	67.5
	2	.389	1.08	.407	7.27	.415	14.9	.447	37.6	.450	121
	3	.322	1.52	.383	5.27	.365	19.3	.391	40.3	.382	110

Final Remarks

Contribution:

- Investigation of encodings for core computations:
  Analysis and encoding of needed properties in SAT/SMT.
- Benchmarks:
  Trade-off between readability and performance.

Tool support:

- CoReS:
  Automatically compute core graphs via SAT/SMT encodings.

Features:

- GUI mode for visualized core computations.
- Integrable and executable standalone command line interface.
- User-manual and source code (Python) available on GitHub:
  `https://github.com/mnederkorn/CoReS`

Thank You

for your attention

Part IV

Additional Material

Invariant checking

Closure under Rewriting

Question: Given T and a (DPO) GTS rule $r = (L \leftarrow I \rightarrow R)$.
Does $Post_{\{r\}}(\mathcal{L}(T)) \subseteq \mathcal{L}(T)$ hold?

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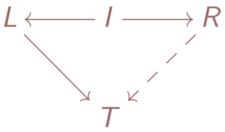
Invariant checking

Closure under Rewriting

Question: Given T and a (DPO) GTS rule $r = (L \leftarrow I \rightarrow R)$.
Does $Post_{\{r\}}(\mathcal{L}(T)) \subseteq \mathcal{L}(T)$ hold?

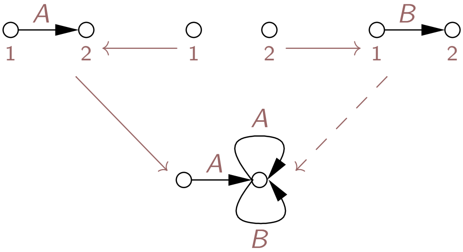
$Post_{\{r\}}(\mathcal{L}(T))$ can not be computed...

Sufficient condition: Check whether for each morphism $L \rightarrow T$ there exists a morphism $R \rightarrow T$ such that the diagram below commutes. This implies closure under rewriting.



The missing piece

This is **not** an if-and-only-if condition. Counterexample:



However, the type graph represents *all* graphs with A- and B-labelled edges and is hence closed under rewriting.

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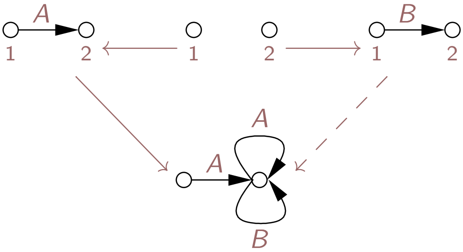

However, the type graph represents *all* graphs with A- and B-labelled edges and is hence closed under rewriting.

Solution: We obtain an if-and-only-if condition if we require that the type graph T is a core!

Experiments

Additional SAT runtimes

		$\rho \cdot \lvert V \rvert \cdot \lvert \Lambda \rvert$										
$\lvert V \rvert$	$\lvert \Lambda \rvert$	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
	1	.462	.595	.309	.333	.351	.359	.388	.476	.371	.589	.354
24	2	.337	.356	.548	.587	1.29	.623	.685	.685	.511	.739	.497
	3	.410	.401	1.00	.460	.456	.871	.450	.490	1.60	.615	.574
	1	.619	.828	.901	1.17	1.11	.85	.973	1.29	.986	1.01	1.53
32	2	.683	.809	.792	.988	1.03	1.27	1.04	1.13	1.23	1.22	1.23
	3	1.13	1.01	.821	.819	1.16	.937	1.10	1.05	1.87	1.27	1.20
	1	2.39	2.62	3.27	3.15	4.45	5.18	5.34	7.18	5.01	5.93	6.24
48	2	1.83	1.83	3.23	3.68	3.97	3.98	4.75	5.47	4.98	5.02	5.37
	3	2.35	2.57	3.06	3.25	3.59	3.94	3.88	4.17	4.28	5.33	4.96
	1	6.63	8.65	12.0	12.7	19.4	21.9	21.2	26.2	22.5	22.1	26.0
64	2	4.04	5.91	6.73	10.9	10.3	14.9	15.2	15.2	15.4	15.7	18.4
	3	4.53	5.60	7.22	8.96	9.02	11.0	10.6	12.0	12.7	11.9	12.1
	1	37.5	49.8	92.8	125	123	165	140	163	193	152	194
96	2	28.6	49.9	59.7	85.5	98.9	102	107	115	127	111	116
	3	23.7	36.7	50.4	60.6	52.0	51.8	48.8	52.6	49.0	44.0	46.6